Team No. 449

Problem A

Volcanism on Super-Earths

Abstract

In this problem, we are required to predict the volcanic activities on the super-earths. To solve the problem, we set up a model based on the conservation of energy, the heat conduction law and the decay law of isotopes. Then we give the simulation of the volcanic activities using MATLAB and Mathematica, and the results show an approximated relationship between the volcanic activities and the mass of the super-earth clearly. After optimizing the model and discussing the strengths and weaknesses, we derived the conclusion: the volcanic activities become intenser gradually at the beginning and slow down after some time. The larger mass a super-earth possesses, the longer the volcanic activities can last.

1 Introduction

In the universe, there are many planets which are very similar to our Earth by structure and components. However, their mass may be different from the Earth's mass, for example, ranging from 0.5 to 3 times of the mass of Earth. These planets are called "super-earth", and the volcanic activities on the planets are of vital importance because it can help people explore the formation and evolution of these planets. This article gives a method to predict the volcanic activities on a super-earth.

In the article, we start from the earth and set up a model: we define the volcanic activity as a quantity proportional to the energy stored in the lava during a certain period of time, analyze the energy gain and loss of the earth, and use the conservation of energy to obtain the energy transferred to the lava. When the temperature of the lava is lower than 700 ℃(the lower bound of temperature interval of the lava),the volcanic activity is regarded as stopped. Combining all together we finally derive a differential equation and solved it using MATLAB and Mathematica. After getting the numerical solutions and expanding it to all the super-earths with dimensional analysis. we get the conclusion: the larger radius a super-earth has, the longer volcanic activities can last on it.

2 Model

2.1 Assumptions

In this problem, we first consider the super-earth as a planet having the same structure as the Earth. As shown in figure 2.1, the super-earths consist of four parts: inner core, outer core, mantle, crust. Between the mantle and the crust, we assume that the super-earths have a layer of viscous lava. Here we assume that the thickness of the layer of lava is trivial compared to the thickness of the crust. This is reasonable, since the thickness of the layer of lava is just several hundred meters. [1] Since we do not have any idea about how the density and the heat capacity change with temperature and pressure, it is very difficult to find the functions $\rho(r)$ of the super-earths, so we assume that the density and specific heat capacity of each parts are constant.

As seen in the figure 2.3.2 [2], the geothermal curve can be regarded as a piecewise linear function. This is quite reasonable. We will give the reason in latter section.

Since we consider a large scale of time, that this, in the scale of 10^8 year. The time between two volcanic eruptions is too small so that we can regard the eruption as a continuous process.

Since the earthquake and volcanic eruption are both the energy release of layer of lava, we do not distinguish them. We think that they are at least in proportion. We denote the energy release of volcano as E_{vol} and the total energy release as E_{rel}, by our assumption, we have

$$E_{vol} = \lambda_{vr} E_{rel}$$

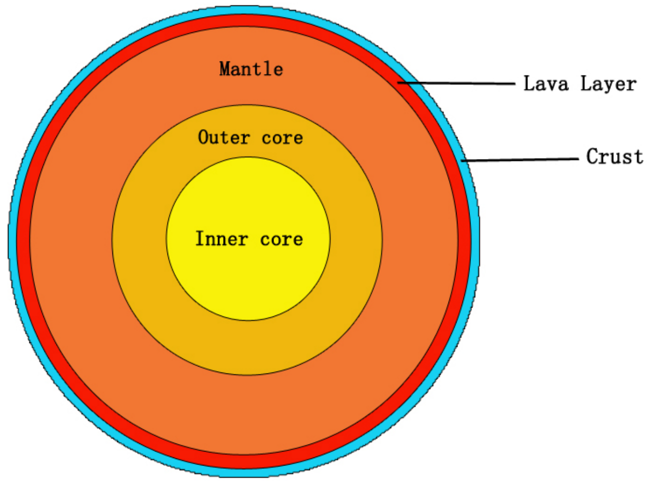

Figure 1: The Structure of Super-Earth (Not to Scale)

We call λ_{vr} volcanic eruption factor. Since this is just small linear constant, it is trivial for our model. Here, we set $\lambda_{vr} = 1$. That is, all the energy release becomes the volcanic eruption.

We assume that the temperature distribution in each part of the super-earth is linear.

We also assume that the thermal conductivity in the same part of the super-earth stays the same.

2.2 Model Overview

The volcanic activity is related to the temperature of the inner part of the planet. In order to give relation of the volcanic activity over time, we need to find the volcanic activity varies over time. Our model mainly focuses on a universal law in physics, *Conservation of the Energy*. Using this law, we get the equation (3), in words, the energy for volcanic eruption equals the difference between the energy transfered to the layer of lava and the energy released from the inner part of the planet received on the surface of the planet.

2.3 Basic Model

We first define the volcanic activity as the energy release of the layer of lava. Since the core of the super-earth releases energy due to the decay of heavy elements and the residual heat from the early age of the super-earth, it heats the layer of lava. The layer of lava also releases the energy to the crust, however, it also stores the energy by various of method [3]. When the storage of the energy reaches its maximum capacity, the layer of lava releases the energy. By our assumption, λ_{vr} of this kind of energy release leads to volcanic activities. Denote the energy stored by the layer of lava as E_{store}, we have

$$E_{vol} = \lambda_{vr} E_{store}$$

By conservation of energy, we have

$$E_{trans} - E_{received} = E_{store} \tag{1}$$

where $E_{received}$ stands for the energy release by thermal transfer (including thermal conduction and thermal radiation) detected at the surface of the super-earth and E_{trans} stands for the energy transfer from mantle of the super-earth to the layer of lava.

Denote ΔU as the change of internal energy of the super-earth and E_{decay} as the energy supply by decay of heavy elements. By conservation of energy, we have

$$\Delta U = E_{decay} - E_{trans} \tag{2}$$

By (1) and (2), we have

$$E_{store} = E_{decay} - \Delta U - E_{received}$$

Take the partial differentiation by time on the both side, we have

$$\frac{\partial E_{store}}{\partial t} = \frac{\partial E_{decay}}{\partial t} - \frac{\partial \Delta U}{\partial t} - \frac{\partial E_{received}}{\partial t}$$

So finally, we have

$$\frac{\partial E_{vol}}{\partial t} = \lambda_{vr}(E_{trans} - E_{received}) = \lambda_{vr}(\frac{\partial E_{decay}}{\partial t} - \frac{\partial \Delta U}{\partial t} - \frac{\partial E_{received}}{\partial t}) \tag{3}$$

2.3.1 Decay Energy

The core of the Earth has huge magnitude of radioactive isotopes at its early age. *University Physics* shows that the decay of the ^{238}U supplies the energy for the core.[5] So we have

$$\frac{\partial E_{decay}}{\partial t} = U_{238U} e^{-\frac{-t}{\tau_{238U}}}$$

where U_{238U} is the initial radiance of ^{238}U and τ_{238U} is the time constant of τ_{238U}. The research shows that $U_{238U} = 20 \cdot 10^{12} W$.[6] Since in our assumption we assume that the super-earths has the same structure of the Earth, we can easily calculate $\frac{\partial E_{decay}}{\partial t}$ on the super-earth.

2.3.2 Heat Transfer

Here we regard mantle, inner core and outer core as a whole and calculate its internal energy. Note that the internal energy of an object can be calculated as

$$U(r) = \iiint_{\Sigma} C(r)T(r)\rho(r)\,\mathrm{d}V$$

where Σ is the sphere, $C(r)$ is the specific heat capacity function, $\rho(r)$ is the density function and $T(r)$ is the temperature function.

Shown in the figure 2.3.2, by the research result [2], we have the thermal distribution (temperature distribution) of the Earth.

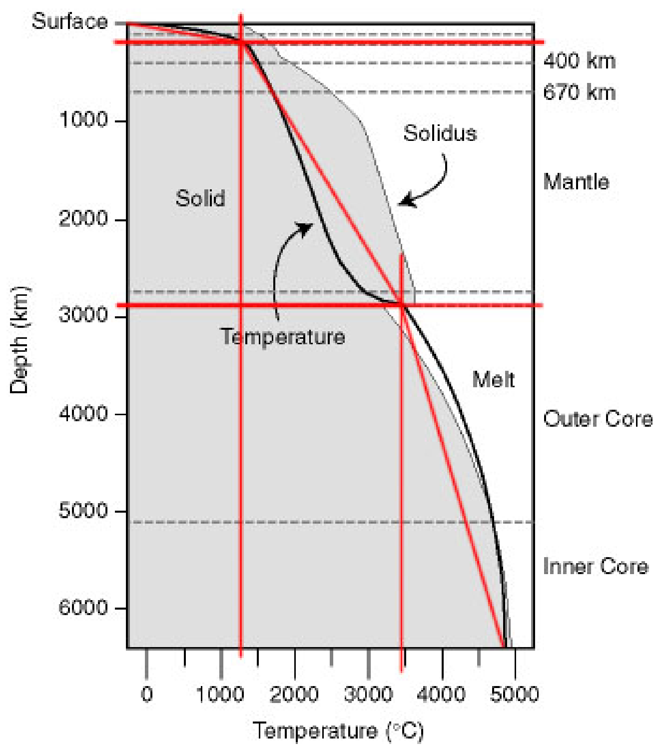

Figure 2: The Geothermal Curve of the Earth

As we discussed before, we regard the geothermal curve of the Earth as piecewise linear function. We make this approximation since the T is linear to U and the relative error is quite small (see figure 2.3.2). Another reason is that this kind of geothermal distribution is just a kind of approximation. It is evident that we human beings lack a direct way to measure these temperatures. If the geothermal function is piecewise function, we can apply the Fourier's law of heat conduction without knowing thermal conductivity.

$$T = T_0 + \frac{T - T_0}{L}$$

where T_0 indicates the known temperature and L denotes the distance between the points whose temperature is T and T_0.

Since we suppose the super-earths have the same structure as the Earth and the geothermal curve is piecewise linear, we can calculate its internal energy by simply multiplying a factor $\lambda_R(r)$. By the Fourier's law of heat conduction, the geothermal function of the Earth obtained directly by is given as following.

$$T(r) = \begin{cases} T_1 - \frac{2(T_1-T_2)}{R} r & \text{if } 0 \leq r < 3000 \text{ m}, \\ T_2 - \frac{2(T_2-T_3)}{R}(r - \frac{1}{2}R) & \text{if } 3000 < r \leq (6370 - 100) \text{ m}, \end{cases} \tag{4}$$

where $T_1 = 5000\text{K}$ is the temperature of center of inner core, $T_2 = 3500\text{K}$ is the temperature of the surface between inner core and outer core, and $T_3 = 1300\text{K}$ is the temperature of the surface between outer core and mantle.

By the assumption, we can obtain the density function of the Earth

$$\rho(r) = \begin{cases} 4.0 \cdot \ \text{kg} \cdot \text{m}^3 & \text{if } 0 \leq r < 3000 \text{ m}, \\ 12 \cdot \ \text{kg} \cdot \text{m}^3 & \text{if } 3000 < r \leq (6370 - 100) \text{ m}, \end{cases} \tag{5}$$

Note that (5) only gives the density of the core and the mantle.

By our assumption, we can get the specific heat capacity function of the Earth

$$C(r) = \begin{cases} 477.3 \text{ J} \cdot \text{kg}^{-1} \cdot \text{K}^{-1} & \text{if } 0 \leq r < 3000 \text{ m}, \\ 1500 \text{ J} \cdot \text{kg}^{-1} \cdot \text{K}^{-1} & \text{if } 3000 < r \leq (6370 - 100) \text{ m}, \end{cases} \tag{6}$$

Substitute (5), (4), (6), by numerical calculation, we have

$$U(r) = U(r) = U(r) = \iiint_\Sigma C(r)T(r)\rho(r)\,\mathrm{d}V \tag{7}$$

As for $\frac{\partial E_{vol}}{\partial t}$, by the Fourier's law of heat conduction, we have

$$\frac{\partial E_{vol}}{\partial t} = \frac{4\pi r^2 \lambda (T - T_0)}{d} = B(T - T_0) \tag{8}$$

where $\lambda \approx 0.025\text{W/m}^2$[7], $d \approx 150\text{m}$.[8]. So we can get $B = 8.5 \cdot 10^1 0\text{W/K}$.

2.3.3 Core Energy Received on Surface

Basalt is the main component of the crust. The experience in the daily life and the contents on the textbook [4] shows that the basalt has excellent property of thermal insulation. Since the research shows that the average thickness of the crust of the Earth is 39 km.[4] It is reasonable that we can omit the energy release by the heat conduction by the thick crust. Even if in the super-earth which the mess is 0.5 times of the earth, the thickness of the crust is

$$d = d_0 \cdot \sqrt[3]{0.5} = 31\text{km}$$

which is also very thick and has excellent heat insulation. Therefore, We can regard that all the energy store in the layer of lava and release in the volcano eruptions or earthquakes. So, we have

$$E_{received} = 0$$

One the one hand, since the heat insulation of the crust is so great we can consider that the energy obtained from the "sun" (the star in the corresponding "solar" systems) will not be conducted to the inner part of the planet. On the other hand, as for the Earth, the thermal radiation received from the Sun equals to the thermal radiation released by the Earth surface. As for super-earth, we know nothing about its sun, so we can assume that this relation also holds.

2.3.4 Volcanic Activity Rate

We define the level of volcanic activity as the rate η:

$$\eta = \frac{\Delta N}{\Delta t}$$

where ΔN is the times of the eruption in Δt, Q is the total energy released in an single eruption. By the assumption, we have the following approximation

$$\frac{\Delta N}{\Delta t} = \frac{\Delta E_{vol}}{Q\Delta t} = \frac{\Delta E_{vol}}{\Delta t} \cdot \frac{1}{Q} = \frac{1}{Q}\frac{\mathrm{d}E_{vol}}{\mathrm{d}t} = \frac{P_{vol}}{Q}$$

So, we have

$$\eta = \frac{P_{vol}}{Q}$$

2.4 Extended Model

Another research shows that ^{238}U is not the only radioactive isotopes which supplies the Earth energy. The Earth are mainly heated by the decay of ^{40}K, ^{235}U, ^{238}U and ^{232}Th.As seen in the figure 2.4, we can calculate E_{decay} precisely. We can improve our model by considering these energy of decay. So, we have

$$\frac{\partial E_{decay}}{\partial t} = U_{40K}e^{-\frac{-t+T_a}{\tau_{40K}}} + U_{235_U}e^{-\frac{-t+T_a}{\tau_{235U}}} + U_{238U}e^{-\frac{-t+T_a}{\tau_{238U}}} + U_{232Th}e^{-\frac{-t+T_a}{\tau_{232Th}}} \quad (9)$$

where T_a indicates the age of the Earth (4.5 billion years), U_{40K}, U_{235_U}, U_{238U} and U_{232Th} denotes the initial radiance of ^{40}K, ^{235}U, ^{238}U and ^{232}Th respectively.

3 Results

3.1 The Numerical Formula

Set $p = \frac{r}{R_0}$ as the ratio of the super-earth and the Earth (R_0 is the radius of the Earth), we can get the $T(p,t)$ from the equation:

$$T'(p,t) = 2 \cdot 10^{13}\mathrm{Exp}\left[-t\big/\left(1.9 \cdot 10^{17}\right)\right] - 8.5 \cdot 10^{10} \cdot (T[t] - 14)/p, T(p,0) = 1300 \quad (10)$$

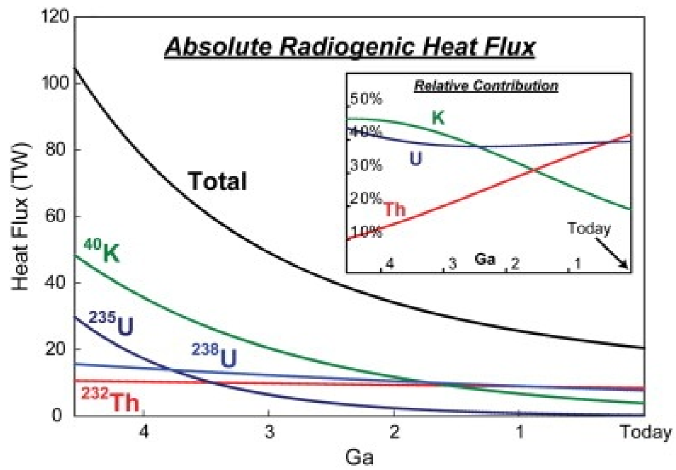

Figure 3: Radiogenic Heat Production for the Four Most Important Isotopes

The equation is just an inhomogeneous ordinary equation of first order. We can simple get the solution:

$$T(p,t) = \frac{(1.19 \times 10^{12} + e^{-\frac{8.5\times 10^{10} t}{p}}(1.09 \times 10^{14} - 2.0 \times 10^{13} p) - 7.37 \cdot 10^{-17} p + 2.0 \cdot 10^{13} e^{-5.26 \cdot 10^{-18} t} p}{8.5 \times 10^{10} - 5.26 \cdot 10^{-18} p} \tag{11}$$

where $T(p,t)$ is the temperature of the outer surface of the mantle of the super-earth with the radius of pR_0 at time t. Note that we denote the present by $t = 0$.

The power of energy obtained by the layer of lava is given as

$$P_{vol} = \frac{\partial E_{vol}}{\partial t} = (T(p,t) - 700) \cdot 8.5 \cdot 10^{10} \tag{12}$$

3.2 Simulation of Basic Model

We denote $\xi = \frac{m}{m_0}$ as the ratio of the mess of the super-earth and the mess of the Earth. We have

$$\xi = p^3$$

We use p instead of ξ in the formula (10) and (11) since p can make the formulae simpler.

Set $\xi = 0.5,\ 0.066$(Mercury), 1(Earth), 3.0. From figure (3.2), we can see that if these "super-earth" are in the same status of the Earth now, we can see a very interesting result. By our model, the Mercury will only have $5 \cdot 10^{16} s = 1.5$ billion years to finish its volcanism. As for the Earth, it has $1.6 \cdot 10^{17} s = 5.07$ billion years, which is quite sound. As for given super-earth, the activities last from $1.1 \cdot 10^{17} s$ to $2.5 \cdot 10^{17} s$ with $\xi = 0.5$ to 3. From here we can see that our model is quite reliable.

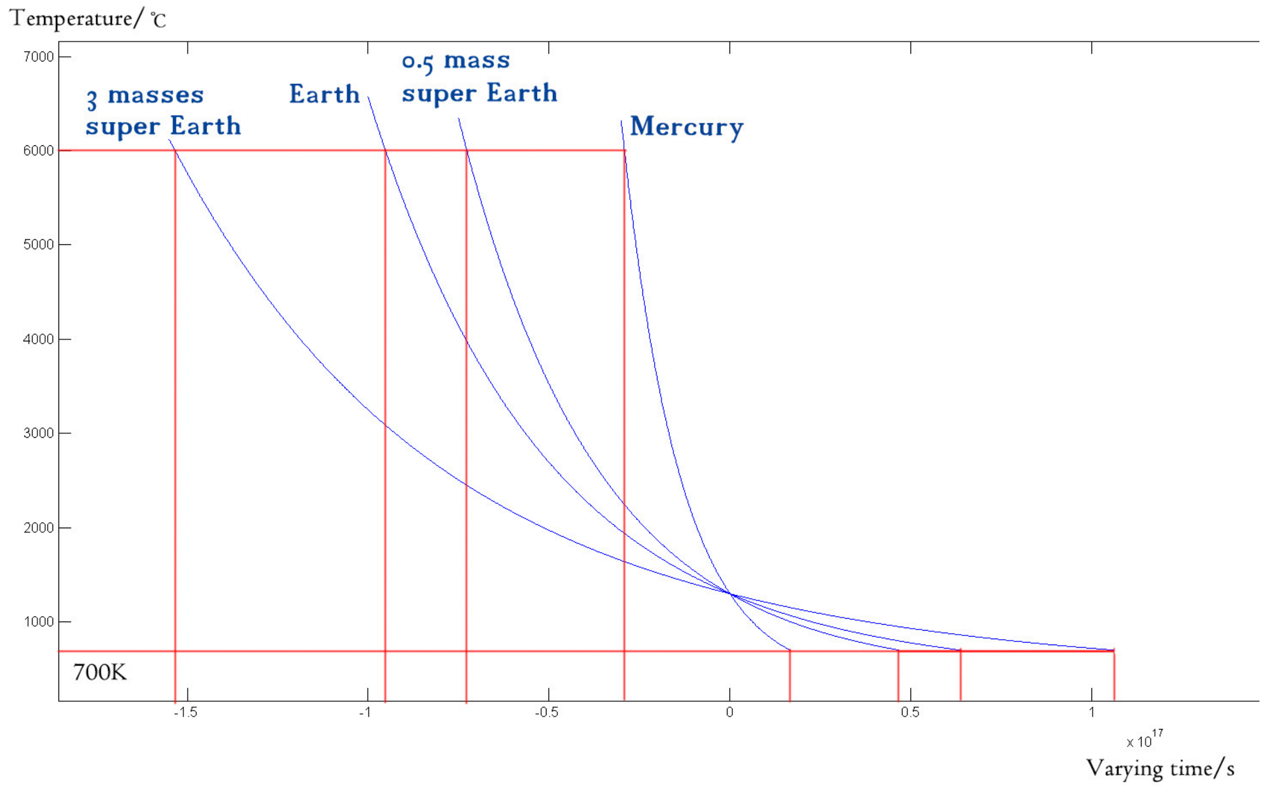

Figure 4: The Temperature Curve of Outer Surface of Mantle of Different Super-Earths

Assume the following super-earths have the same initial temperature. As seen in the figure (3.2), here we set $p = 0.7$ to 1.4 by step 0.1 (this equals to $\xi = 0.5$ to 3). We can see that all the curves act as exponential decay. We can also see that the smallest super-earth has the life of volcanic activities $5.5 \cdot 10^{16}s = 1.74$ billion years shorter than the Earth. As for the biggest super-earth, it has the life of the volcanic activities $1.5 \cdot 10^{17}s = 4.76$ billion years longer than the Earth. These results gives a good prediction of volcanic activities of different super-earth varies by the time.

3.3 Simulation of Extend Model

After calculating the preciser E_{decay}, we can get a new set of figures. We can see that the new model indicates a new feature, which is very interesting.(See figure 3.3) The temperature of the outer surface of mantle increase at first and decrease at last. This shows that the temperature of the surface of the Earth was very low at first, then due to the decay of the isotopes, the Earth was been heated and then its temperature rise. When the mass of the isotopes decrease as well as the radiance of the isotopes decrease, the temperature goes down. So the level of volcanic activities also increase at first and

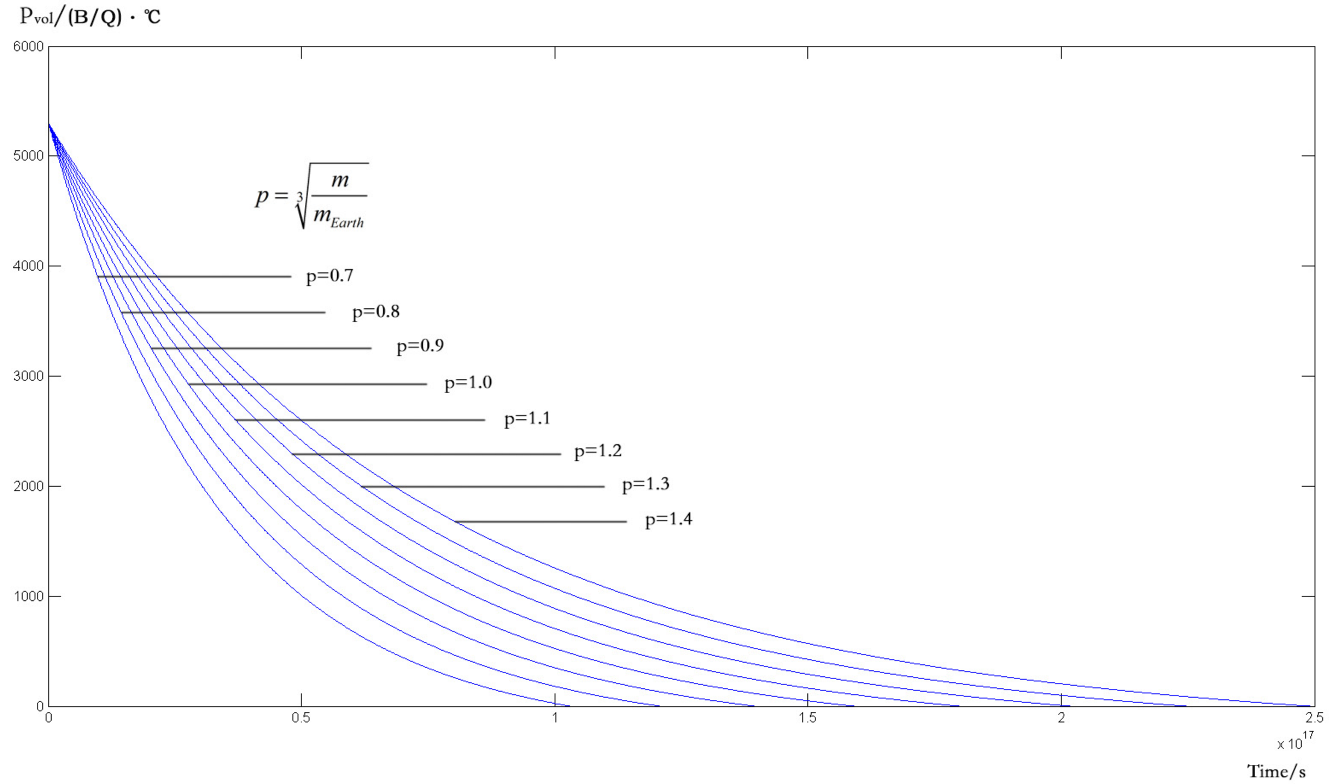

Figure 5: The Volcanic Activity Curve varies by time of Different Super-Earths

finally go down. We consider this is quite reasonable.

As for $\xi = 0.5$, $\xi = 0.73$, $\xi = 1.73$, $\xi = 2.74$, we can see that the larger the super-earth. the later it starts the temperature decrease. This is quite sound, since larger super-earth has longer volcanic activities period.

4 Discussion

4.1 Weaknesses

First, we lack too many conditions and details. The whole problem is set up on a large scale of time and space, and there is little information about the thermal history of the Hadean and Archean Eon as well as about the interior circumstances of the Earth. For example, we can't find any information about the thickness of the lava layer, which is important in our formula. Therefore we have to estimate some parameters and get a possible solution. When calculating the interior energy of the Earth, though we have the temperature-depth relationship of the Earth, we can't find the specific heat capacity of iron and silicate with such pressure and temperature, so we have to replace them by

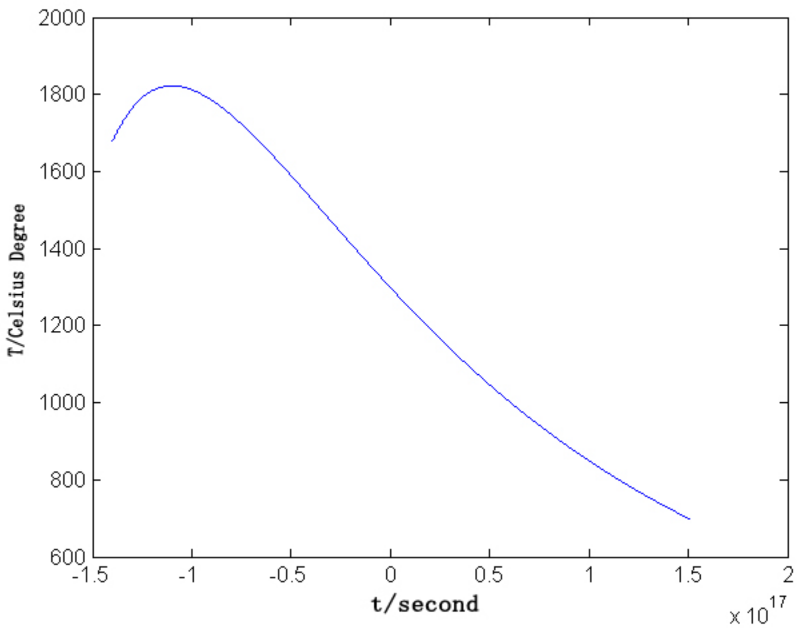

Figure 6: The Temperature of Outer Surface of Mantle of the Earth in the Extended Model Varies by Time

approximated values.These estimations and approximations can cause considerable error.

Second, we make too much simplification compared with the original complicated situation. The volcanic activity is a series of very complex physical-chemical reaction but we just use such simple conditions and physical laws to simulate, so there are many processes influencing the results that we have neglected. However, this is the best model we can formulate using all the information we have.

Finally, during the simulation of the evolution, since the time interval is too large, we have to set 10^{13} seconds($3 \cdot 10^5$ year) as a step so that the computer can accomplish the work. This large step may also cause errors.

4.2 Strengths

First, our model is simple and directly to perceive, it can show clearly a trend of the change of the volcanic activity with the change of the radius. From the result we can see the model fits the real situations well, especially when we consider the radiogenic emitted from different elements. Therefore we can use it to predict the volcanic activity on different super-earths with time varies approximately.

Second, our model is based on a series of reasonable assumptions. We assume the energy released from volcano eruption is proportional to that released from earthquakes, considering the volcanoes and the seismic zones distributes similarly, thus simplifies the calculation. We also choose to neglect the heat flow from the earth, because the isolation of the crust is excellent, meanwhile the solar radiance is also providing energy and the surface of the earth has been in thermal balance for billions of years.

Furthermore, our model is based on the real situation of the energy flow. We referred the documentations and specified that the two main energy sources are the residual heat and the radiogenic heat. This can help to set up a clear equation and make the model more valid than just assuming.

Generally speaking, it is very difficult to analyze such a large and complicated process with so little information, but we did it. Though this model is kind of rough, we show a satisfying trend of the volcanic activity successful. This is the biggest victory of the modeling.

4.3 The Dependence of the Results on Parameters

Since we estimate a lot of parameters due to a lack of information, now we try to analyse the dependence of the results on them.

We choose coefficient B in formula (8) because it contains two estimated parameters: the conductivity λ and the thickness of the lava layer d. During modeling, we didn't find exact information on λ and d, so we just estimate them as $0.025\mathrm{W/m}^2$ and d as 150m on average. However, consider if $\lambda = 0.03\mathrm{W/m}^2$ and $d = 100$m (it's very likely), then $B = 1.53 \cdot 10^1 1W/K$. We tried the simulation again and find that the shape of the curve as well as the time period before the volcanoes die will be much smaller. Actually, it is just like we are considering an Earth-like planet with radius $r = 0.55R$, where R is the radius of the Earth. Therefore, we can see that the estimation of the conductivity and the thickness of lava layer will severely influence the result.

References

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Appendix A: Plot of the Figures

```
i=0;
tt = 1300 ;
T_ar = [tt];
T = tt;
t_ar =[0];
t = 0;
dt = 10^13;
p = 1;

while i<14191
    T = T - (2*10^13)*exp(-t/1.9*10^17)-8.5*10^10*(T-14)/p)*dt/(4*10^27);
    t = t - dt;
    T_ar = [T T_ar];
    t_ar = [t t_ar];
    i = i+1;
end
i=0;
T_arr = [tt];
T = tt;
t_arr =[0];
t = 0;


while i<30000
    T = T + (2*10^13)*exp(-t/1.9*10^17)-8.5*10^10*(T-14)/p)*dt/(4*10^27);
    t = t + dt;
    T_arr = [T_arr T];
    t_arr = [t_arr t];
    i = i+1;
    if T<700
        break
```

```
    end
end
T_ar = [T_ar T_arr];
t_ar = [t_ar t_arr];
plot(t_ar,T_ar)
```

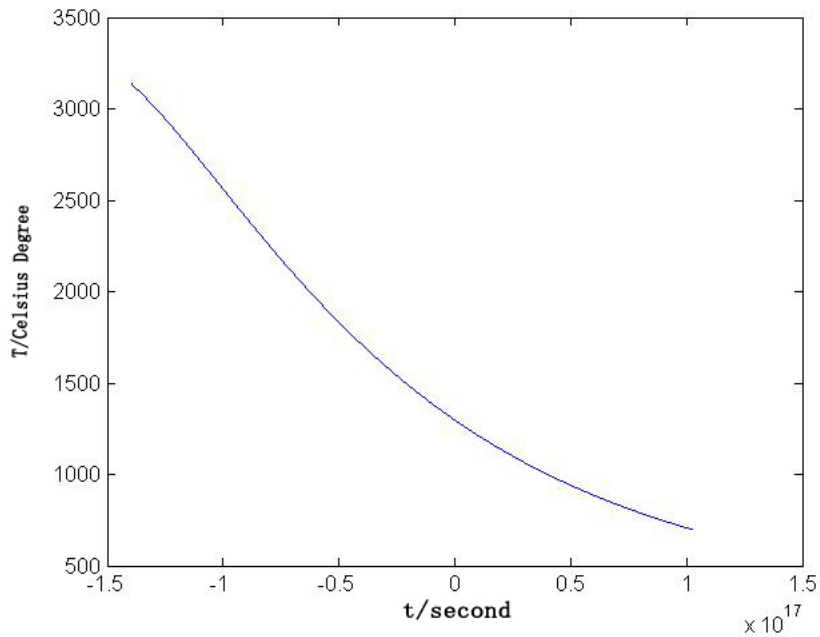

Figure 7: The Temperature of Outer Surface of Mantle of the super-earth ($\xi = 0.5$) in the Extended Model Varies by Time

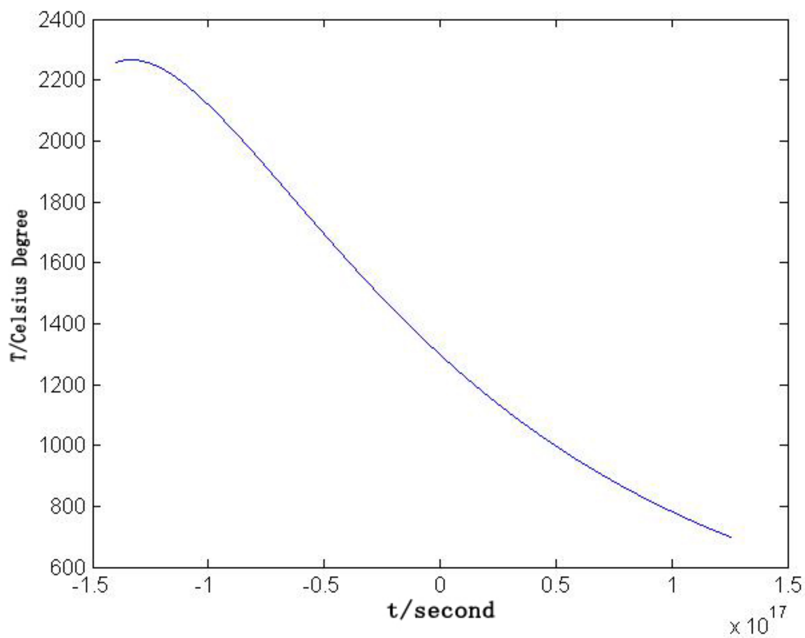

Figure 8: The Temperature of Outer Surface of Mantle of the super-earth ($\xi = 0.73$) in the Extended Model Varies by Time

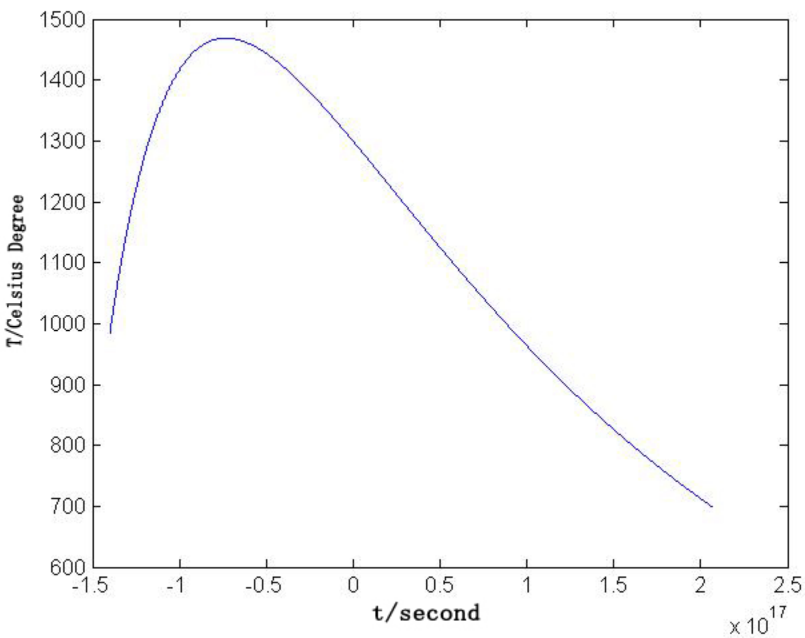

Figure 9: The Temperature of Outer Surface of Mantle of the super-earth ($\xi = 1.73$) in the Extended Model Varies by Time

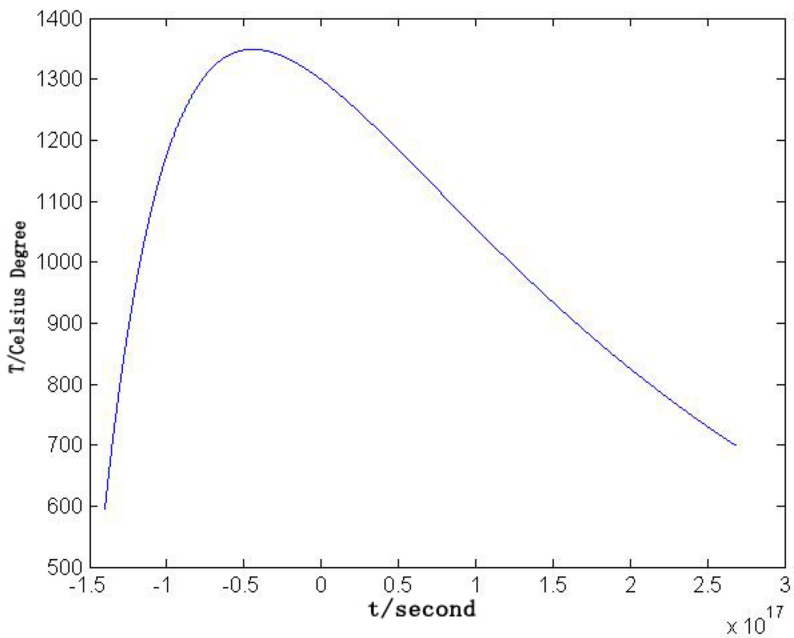

Figure 10: The Temperature of Outer Surface of Mantle of the super-earth ($\xi = 2.74$) in the Extended Model Varies by Time